$$L = 1/2[(1-2m/r)c^2\dot{t}^2 - (1-2m/r)^{-1}\dot{r}^2 - r^2 \dot{d\phi}^2] \qquad [1 \text{ mark}]$$

the Euler Lagrange equations for cyclic coordinates eg t is

$$\frac{d}{d\tau}\Big(\frac{\partial L}{\partial \dot{t}}\Big) - \frac{\partial L}{\partial t} = 0 = \frac{d}{d\tau}\Big(\frac{\partial L}{\partial \dot{t}}\Big)$$

$$(1-2m/r)\dot{t} = k$$

where k is a constant. A similar analysis for ϕ gives $r^2\dot{\phi} = h$ where h is a constant. [1 mark]

substitute for both $\dot{t}$ and $\dot{\phi}$ to get $\dot{r}$ from the null metric so

$$(1-2m/r)c^2\dot{t}^2 - (1-2m/r)^{-1}\dot{r}^2 - r^2\dot{\phi}^2 = 0 \qquad [1 \text{ mark}]$$

$$(1-2m/r)c^2\frac{k^2}{(1-2m/r)^2} - (1-2m/r)^{-1}\dot{r}^2 - r^2\frac{h^2}{r^4} = 0 \quad [1 \text{ mark}]$$

$$c^2k^2 - \dot{r}^2 - h^2(1-2m/r)/r^2 = 0$$

$$\dot{r}^2 = c^2k^2 - h^2(1-2m/r)/r^2 = c^2k^2 - h^2V^2(r) \qquad [1 \text{ mark}]$$

maximum at

$$dV^2/dr = 0 = -2(1-2m/r)/r^3 + 1/r^2 d(1-2m/r)/dr = -2(1-2m/r)/r^3 + 1/r^2 2m/r^2$$

so at $-2 + 4m/r + 2m/r = 0$ i.e. $r = 3m$ [1 mark]

The maxium value is $V^2(3m) = (1-2/3)1/(9m^2) = 1/(27m^2)$ [1 mark]

null geodesics have

$$\Big(\frac{du}{d\phi}\Big)^2 + u^2 = \frac{c^2k^2}{h^2} + 2mu^3$$

For $m \to 0$ this goes to $(du/d\phi)^2 + u^2 = c^2k^2/h^2$. Substitute in $u = u_0 \sin\phi$ and this gives $u_0^2\cos^2\phi + u_0^2\sin^2\phi = c^2k^2/h^2$ so $u_0 = ck/h$ or $b = 1/u_0 = h/ck$ as the impact parameter [1 mark]

$\dot{r}^2 > 0$ means light can get everywhere (including down the black hole, where it is captured). So capture when $c^2k^2 - h^2V^2(r) > 0$ ie $V^2(r) < c^2k^2/h^2$ but $b = h/ck$ so this is $V^2(r) < 1/b^2$. [1 mark]

But from above $V^2 \le 1/(27m^2)$, so if $b \le \sqrt{27}m$ then the light is captured. [1 mark]