(a)

$$E_1^1 = \int \psi_{100}^{0*} H' \psi_{100}^0 dx = \frac{1}{\pi a^3} \int\int\int e^{-r/a} - eE_{ext} r \cos\theta e^{-r/a} r^2 \sin\theta d\theta d\phi$$

$$= \frac{-eE_{ext}}{\pi a^3} \int\int e^{-2r/a} r^3 \int_0^\pi \cos\theta \sin\theta d\theta dr d\phi$$

$$\text{but} \quad \int_0^\pi \cos\theta \sin\theta d\theta = [-\frac{1}{2}\cos^2\theta]_0^\pi = -\frac{1}{2}\cos^2 0 + \frac{1}{2}\cos^2\pi = 0$$

so there is no first order correction

(b: 200)

$$\int \psi_{200}^{0*} H' \psi_{100}^0 dV = -eE_{ext} \int\int\int \psi_{200}^* r \cos\theta \psi_{100} r^2 \sin\theta dr d\theta d\phi$$

neither ψ_{200}^{0*} nor ψ_{100} has any θ dependence so the θ integral is $\int_0^\pi \cos\theta \sin\theta d\theta = 0$ as before so there is no contribution from this term.

(b: 211) the θ integral of $\psi_{211}^{0*} H' \psi_{100}$ is

$$\int_0^\pi \sin\theta \cos\theta \sin\theta d\theta = [\frac{1}{3}\sin^3\theta]_0^\pi = 0$$

so there is no contribution from this either.

(b: 21-1) same integral as above for ψ_{211}.

(b: 210) the θ integral for $\int \psi_{210}^{0*} H' \psi_{100} dV$ is

$$\int_0^\pi \cos^2\theta \sin\theta d\theta = [-\frac{1}{3}\cos^3\theta]_0^\pi = -\frac{1}{3}(-1)^3 + \frac{1}{3}(1)^3 = \frac{2}{3}$$

so this one we need to calculate in full!

$$\int \psi_{210}^{0*} H' \psi_{100}^0 dV$$

$$= \frac{1}{4\sqrt{2\pi a^3}} \sqrt{\frac{1}{\pi a^3}} \int\int\int \frac{r}{a} e^{-r/2a} - eE_{ext} r \cos\theta e^{-r/a} r^2 \sin\theta dr d\theta d\phi$$

$$= \frac{-eE_{ext}}{4\sqrt{2}} \frac{1}{\pi a^4} 2\pi \frac{2}{3} \int r e^{-r/2a} r e^{-r/a} r^2 dr$$

$$= \frac{-eE_{ext}}{\sqrt{2}} \frac{1}{3a^4} \int e^{-3r/2a} r^4 dr$$

$$= \frac{-eE_{ext}}{\sqrt{2}} \frac{1}{3a^4} \frac{4!}{(3/2a)^5}$$

$$= \frac{-eE_{ext}}{\sqrt{2}} \frac{a}{3} \frac{4.3.2.2^5}{3^5} = \frac{-eE_{ext} a}{\sqrt{2}} \frac{2^8}{3^5} = -\frac{2^7\sqrt{2}}{3^5} eE_{ext} a$$

so the first term of

$$E_1^2 = |\frac{-eE_{ext} a 2^7 \sqrt{2}}{3^5}|^2 \frac{1}{E_1^0 - E_2^0} = \frac{e^2 E_{ext}^2 a^2 2^{15}}{3^{10}} \frac{1}{E_1^0 - E_1^0/4} = \frac{e^2 E_{ext}^2 a^2 2^{17}}{3^{11} E^0}$$

(c)

$$\int \psi^0_{310} H' \psi^{0*}_{100} dV = -eE_{ext} \int\int\int \psi^*_{310} r\cos\theta \psi_{100} r^2 \sin\theta dr d\theta d\phi$$

$$= -eE_{ext} \frac{2}{27}\frac{\sqrt{2}}{\pi a^3} \int\int\int (1-\frac{r}{6a})\frac{r}{a} e^{-r/3a} \cos\theta r \cos\theta e^{-r/a} r^2 \sin\theta dr d\theta d\phi$$

$$= -eE_{ext} \frac{2}{27}\frac{\sqrt{2}}{\pi a^4} 2\pi \int\int (1-\frac{r}{6a}) r^4 e^{-4r/3a} \cos^2\theta \sin\theta dr d\theta$$

$$-eE_{ext} \frac{4}{27}\frac{\sqrt{2}}{a^4}\frac{2}{3} \int (1-\frac{r}{6a}) r^4 e^{-4r/3a} dr$$

$$= -eE_{ext} \frac{8\sqrt{2}}{3^4}\frac{1}{a^4}\Big(\int r^4 e^{-4r/3a} dr - \frac{1}{6a}\int r^5 e^{-4r/3a} dr\Big)$$

$$= -eE_{ext} \frac{8\sqrt{2}}{3^4}\frac{1}{a^4}\Big(\frac{4!}{(4/3a)^5} - \frac{1}{6a}\frac{5!}{(4/3a)^6}\Big)$$

$$= -eE_{ext} \frac{8\sqrt{2}}{3^4}\frac{1}{a^4}\Big(a^5 \frac{4.3.2.3^5}{4^5} - a^5 \frac{5.4.3.2.3^6}{6.4^6}\Big) = -eE_{ext}\frac{8\sqrt{2}}{3^4} a \Big(\frac{3^6}{2^7} - \frac{5.3^6}{2^{10}}\Big)$$

$$= -eE_{ext} \frac{8\sqrt{2}}{3^4} a \Big(\frac{2^3.3^6}{2^{10}} - \frac{5.3^6}{2^{10}}\Big) = -eE_{ext}\frac{8\sqrt{2}}{3^4} a \frac{3^6}{2^{10}}(2^3-5)$$

$$= -eE_{ext} \frac{8\sqrt{2}}{3^4} a \frac{3^6}{2^{10}} 3 = -eE_{ext}\sqrt{2} a \frac{3^3}{2^7}$$

so our second term to the second order correction is

$$\frac{|\int \psi^{0*}_{310} H' \psi^0_{100} dV|^2}{E^0_1 - E^0_3} = e^2 E^2_{ext} 2a^2 \frac{3^6}{2^{14}} \frac{1}{E^0_1 - E^0_1/9} = e^2 E^2_{ext} a^2 \frac{3^6}{2^{13}}\frac{9}{8E^0_1} = e^2 E^2_{ext} a^2 \frac{3^8}{2^{16} E^0_1}$$

so our total is

$$E^2_1 = \frac{e^2 E^2_{ext} a^2}{E^0_1}\Big(\frac{2^{17}}{3^{11}} + \frac{3^8}{2^{16}} + \dots\Big)$$